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LETTER TO THE EDITOR

Universal scaling properties of ballistic deposition and Eden growth on surfaces

Paul Meakin

Central Research and Development Department, E I du Pont de Nemours and Company, Wilmington, DE 19898, USA

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Abstract. The structures generated by ballistic deposition and Eden growth from a surface can be divided in a natural way (defined by the growth process) into 'trees' or clusters of connected sites or particles. The structure of these trees can be characterised by the exponents $\nu_\perp$ and $\nu_\parallel$ which describe how their heights (h) and widths (w) grow with increasing size s ($w \sim s^{\nu_\perp}$, $h \sim s^{\nu_\parallel}$). In addition, the distribution of sizes can be described by the power law $N_s \sim s^{-\tau}$ where N_s is the number of trees of size s. Both the individual trees and the complete deposit are compact so that the exponents $\nu_\perp$, $\nu_\parallel$ and τ satisfy the simple scaling relationships $(d-1)\nu_\perp + \nu_\parallel = 1$ and $\tau = 2 - \nu_\parallel$. The exponents $\nu_\perp$, $\nu_\parallel$ and τ seem to be universal for off-lattice ballistic deposition, on-lattice ballistic deposition and Eden growth models. However, the corresponding exponents for the river network model are different from those for Eden growth and ballistic deposition. From the two-dimensional Eden model and ballistic deposition models values of about 0.40, 0.60 and 1.40 were obtained for $\nu_\perp$, $\nu_\parallel$ and τ, respectively. In three dimensions $\nu_\perp \simeq 0.28$, $\nu_\parallel \simeq 0.46$ and $\tau \simeq 1.54$.

Considerable interest has recently developed in the geometric scaling properties of structures grown by very simple processes such a ballistic deposition (Vold 1959) and Eden growth (Eden 1961). Much of this interest has been stimulated by the realisation that these structures can be described in terms of the concepts of fractal geometry (Mandelbrot 1982) and related geometric scaling relationships. In the case of both the Eden model and ballistic deposition, the internal structure is uniform on all but very short length scales ($D = d$ where D is the fractal dimensionality and d is the Euclidean dimensionality). For growth from a surface or line in strip geometry, the variance in the surface height (ξ) can be described by the scaling form

$$\xi(h) \sim L^\alpha f(h/L^{\alpha/\beta}) \tag{1}$$

first introduced by Family and Vicsek (1985) for ballistic deposition and by Jullien and Botet (1985) for the Eden model. Here h is the mean height above the originally flat surface and L is the width of the strip (or column for $d = 3$). Recent large scale simulations using both ballistic deposition models (Meakin *et al* 1986b) and Eden growth models (Freche *et al* 1985, Hirsch and Wolf 1986, Zabolitsky and Stauffer 1986, Meakin *et al* 1986a, Stauffer and Zabolitzky 1986) are consistent with the idea that the exponents α and β in equation (1) are equal for both models, with α having a value of $\frac{1}{2}$ and β a value of $\frac{1}{3}$ for $d = 2$. These values are predicted by the theoretical work of Kardar *et al* (1986). For $d = 3$ there is considerably more uncertainty but it seems that both models lead to values of about $\frac{1}{3}$ and $\frac{1}{4}$ for α and β, respectively.

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Thus it appears that the self-affine (Mandelbrot 1986, Voss 1986) fractal surfaces of deposits generated by ballistic deposition models and Eden growth models can be described in terms of equation (1) and the universal exponents α and β.

This letter is concerned with a different aspect of the structure of Eden growth and ballistic deposition. In off-lattice ballistic deposition the incoming particles contact only one particle in the growing deposit. Consequently, the structure can be decomposed into trees of connected particles. This is illustrated in figure 1 (see also Meakin 1987). In lattice models for ballistic deposition an occupied site may have more than one nearest neighbour. However, similar trees or clusters can be defined during the growth process. In the two-dimensional simulations carried out in connection with this work, two different models were used. In model I one of the nearest neighbours to a newly deposited site was selected at random and the new site was considered to be part of the same tree as the randomly selected neighbour. In model II the newly deposited site was added to the tree associated with the nearest neighbour which is closest to the basal line at which the growth originates. If the nearest neighbours are all at the same height, one of them is selected randomly and the newly added site is added to the tree associated with that site. In the case of three-dimensional cubic lattice deposition, only model I was investigated.

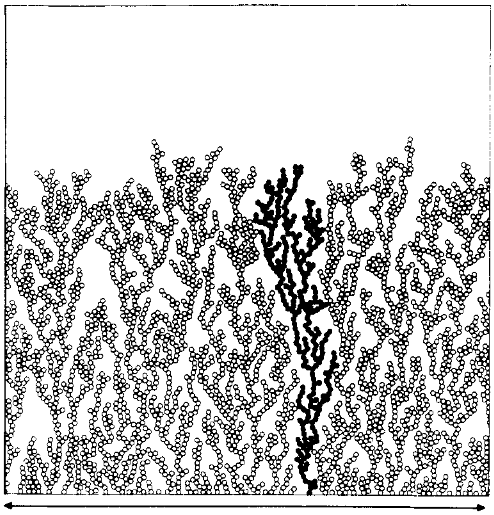

Figure 1. Part of a small-scale two-dimensional simulation of ballistic deposition with normal incidence. One of the 'trees' of connected particles is emphasised by full circles instead of open circles.

Two-dimensional Eden growth was carried out using model 'C' introduced by Jullien and Botet (1985). In this model an occupied surface site is selected at random and one of its unoccupied nearest neighbours is then selected (also randomly) and filled. In this model the newly occupied site was considered to be part of the same tree as the randomly selected occupied surface site.

Three-dimensional simulations were carried out using off-lattice ballistic deposition with spherical particles, all of the same size, following randomly selected trajectories normal to the planar substrate. Simulations were also carried out using a cubic lattice ballistic deposition model analogous to the two-dimensional model I. In this model a newly deposited site is assigned to the cluster or tree to which a randomly selected, occupied, nearest neighbour belongs.

For all of the trees or clusters associated with the models described above, the maximum width (w) and maximum height (h) were measured. For the three-dimensional models the width of a projection onto the xz plane was measured. Here the z direction is perpendicular to the basal surface which lies in the xy plane. The tree size distribution N_s was also measured (N_s is the number of trees containing s sites).

For the 2D off-lattice model 86 simulations were carried out in which 10^7 particles (discs of unit diameter) were deposited onto a base of size $L = 8192$ diameters. Periodic boundary conditions were used in all of the simulations. Figure 2(a) shows the dependence of the maximum height and maximum width of the trees on their sizes (s). The results shown in this figure indicate that

$$w \sim s^{\nu_\perp} \tag{2}$$

and

$$h \sim s^{\nu_\parallel} \tag{3}$$

where the exponents $\nu_\perp$ and $\nu_\parallel$ have values of 0.4 and 0.6, respectively.

Figure 2(b) shows the tree size distribution. It is apparent from this figure that the size distribution can be described by the power law

$$N_s \sim s^{-\tau} \tag{4}$$

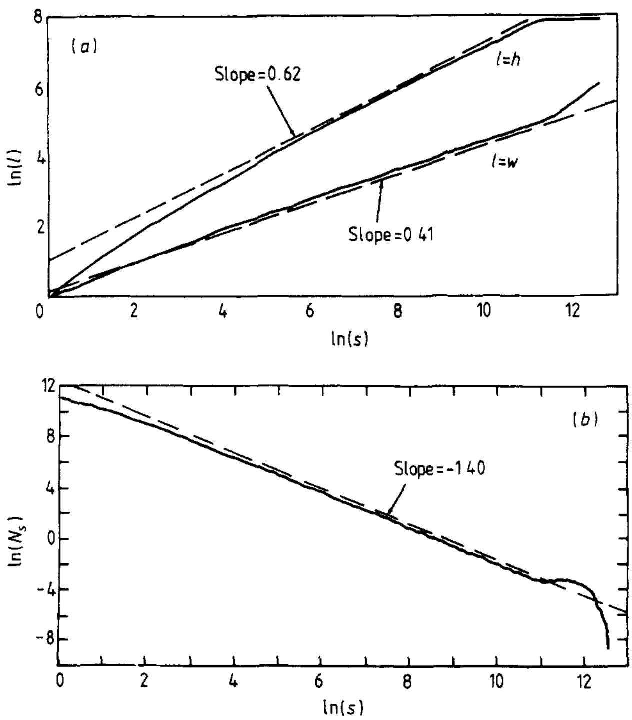

Figure 2. Dependence of (a) the tree height (h) and tree width (w) and (b) number of trees (N_s) on the tree size (number of particles) for two-dimensional off-lattice ballistic deposition. These results were obtained from simulations similar to that illustrated in figure 1 but were carried out on a much larger scale.

for trees which contain more than a few particles but are not large enough to reach the upper surface of the deposit. The exponent τ has a value of about 1.4.

Since the trees are compact on all length scales (the fractal dimensionality D is equal to the Euclidean dimensionality d for ballistic deposition and the trees cannot interpenetrate for $d=2$), the exponents $\nu_\perp$ and $\nu_\parallel$ should satisfy the scaling relationship $\nu_\perp+\nu_\parallel=1$. Using the arguments presented by Racz and Vicsek (1983) for the cluster size distribution exponent τ for DLA, the result $\tau=2-\nu_\parallel$ (compared with equation (3) of Racz and Vicsek (1983)) is obtained.

Using the two-dimensional lattice models for ballistic deposition, more than 10^9 sites were deposited on a base of 2^{18} (262 144) sites. The results from these simulations are shown in table 1. For the two-dimensional Eden model, growth was carried out to a height of 10 000 lattice units on strips of width 2^{14} (16 384) lattice units. The results obtained in table 1 were obtained from 12 simulations with $L=2^{14}$ lattice units.

Table 1. Values for the exponents $\nu_\perp$, $\nu_\parallel$ and τ obtained from the two-dimensional and three-dimensional models. The error limits given in this table represent one standard error in a linear fit of a straight line to the data on a log-log scale. Systematic errors are considerably larger than these statistical contributions to the uncertainty.

Model	$\nu_\perp$	$\nu_\parallel$	τ	$\nu_\perp/\nu_\parallel$
2D models				
Off-lattice ballistic	0.412 ± 0.001	0.616 ± 0.001	1.404 ± 0.004	0.668
Ballistic lattice model I	0.405 ± 0.002	0.610 ± 0.002	1.401 ± 0.006	0.664
Ballistic lattice model II	0.404 ± 0.002	0.604 ± 0.002	1.407 ± 0.006	0.669
Eden	0.401 ± 0.001	0.600 ± 0.001	1.406 ± 0.004	0.668
River network	0.339 ± 0.0004	0.671 ± 0.0004	1.331 ± 0.002	0.505
3D models				
Off-lattice ballistic	0.290 ± 0.002	0.462 ± 0.004	1.573 ± 0.010	0.63
Ballistic lattice model I	0.283 ± 0.002	0.452 ± 0.002	1.566 ± 0.002	0.63
Eden	0.271 ± 0.002	0.465 ± 0.002	1.540 ± 0.003	0.58
River network	0.283 ± 0.001	0.540 ± 0.002	1.462 ± 0.003	0.524

The results shown in table 1 are consistent with the idea that $\nu_\perp+\nu_\parallel=1$ and $\tau=2-\nu_\parallel$ (or $1+\nu_\perp$) for both the two-dimensional Eden model and the two-dimensional ballistic deposition models. These results also strongly suggest that $\nu_\perp$ and $\nu_\parallel$ have the universal values of 0.4 and 0.6, respectively, for two-dimensional Eden growth and ballistic deposition. Kondoh *et al* (1987) have determined the exponents $\nu_\perp$ and $\nu_\parallel$ for Scheidegger's (1967) 'river network' model. In this two-dimensional lattice model nodes at a height Y and position X (horizontal distance) are connected randomly to a node at a height of $Y-1$ and position $X+\frac{1}{2}$ or $X-\frac{1}{2}$. Kondoh *et al* found that $\nu_\perp=\frac{1}{3}$ and $\nu_\parallel=\frac{2}{3}$ for this model and table 1 shows the results obtained from a simulation in which a network with a height of 20 000 lattice units was constructed from a base of 2^{18} lattice units (with periodic boundary conditions). For this model the scaling relationships $\nu_\perp+\nu_\parallel=1$ and $\tau=2-\nu_\parallel$. However, the exponents $\nu_\perp$ and $\nu_\parallel$ are different from those found for the Eden and ballistic deposition models.

Table 1 also shows the results obtained from the three-dimensional models. For the cubic lattice ballistic deposition model deposits were grown to a height of 5000 lattice units in columns of width $L=512$ lattice units. The results shown in table 1 were obtained from 16 such simulations. Figure 3 shows a cross section through a

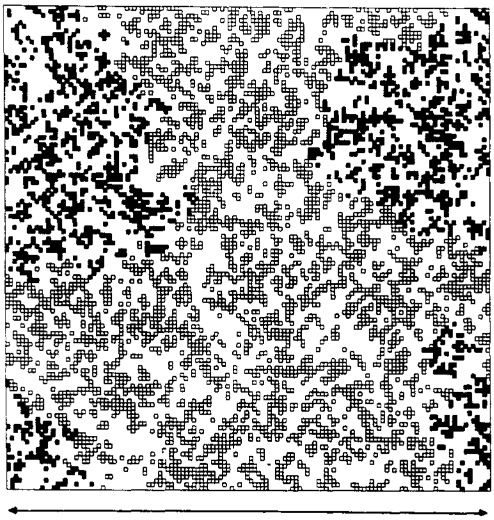

Figure 3. Cross section through a three-dimensional cubic lattice simulation of ballistic deposition. The simulation was carried out in a column of area 128×128 lattice units. The cross section is at a height of 1000 lattice units and the sites belonging to one of the trees are distinguished by full squares instead of open squares.

deposit grown in a column of area 128×128 lattice units at a height of 1000 lattice units. It appears from this figure that the trees are compact mutually excluding structures so that the scaling relationship $2\nu_\perp+\nu_\parallel=1$ is expected. The results shown in table 1 indicate that $\nu_\perp=0.28$ and $\nu_\parallel=0.45$ in accord with this expectation. Figure 4(*a*) shows the dependence of h and w on s for the cubic lattice deposition model and figure 4(*b*) shows the cluster size distribution from which a value of about $1.55=(2-\nu_\parallel)$ was obtained for τ. The off-lattice ballistic deposition model was carried out using columns with an area of 128×128 diameters. From 100 simulations in which 5×10^6 particles were deposited in each simulation, the results shown in table 1 were obtained. These results are consistent with asymptotic values of about 0.29, 0.46 and 1.57 for $\nu_\perp$, $\nu_\parallel$ and τ, respectively.

The three-dimensional Eden model results given in table 1 were obtained using a three-dimensional version of the two-dimensional Eden model 'C' introduced by Jullien and Botet (1985). Sixteen simulations were carried out in which growth was continued to a height of 2500 lattice units in a channel of width 256×256 lattice units. The values of the exponents $\nu_\perp$, $\nu_\parallel$ and τ (approximately 0.27, 0.46 and 1.54) are quite similar to those obtained from both the lattice and off-lattice ballistic deposition models.

Simulations were also carried out using a three-dimensional version of the river network model in which nodes at a height Z and position (X, Y) in the horizontal plane are connected randomly to one of the nodes at height $Z-1$ and position $(X-\frac{1}{2}, Y-\frac{1}{2})$, $(X-\frac{1}{2}, Y+\frac{1}{2})$, $(X+\frac{1}{2}, Y-\frac{1}{2})$ or $(X+\frac{1}{2}, Y+\frac{1}{2})$. In this case the entire network is compact but the individual trees have a non-compact fractal structure (i.e. the trees interpenetrate each other). The results shown in table 1, $\nu_\perp\simeq0.28$, $\nu_\parallel\simeq0.54$, indicate that $2\nu_\perp+\nu_\parallel\simeq1.10$ which is consistent with the observation that the trees are not fully compact. However, it is possible that the asymptotic value of $2\nu_\perp+\nu_\parallel$ is 1.0 since logarithmic (or other) corrections could give rise to an effective exponent of 1.1 with an asymptotic value of 1.0. Similarly, the value of 0.524 for the ratio $\nu_\perp/\nu_\parallel$ is not

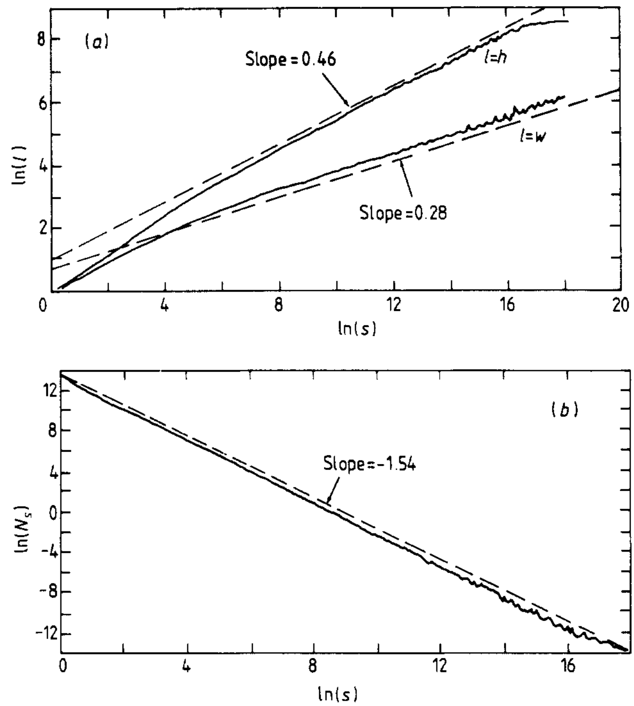

Figure 4. Dependence of the mean cluster height (h), mean cluster width (w) and number of clusters (N_s) obtained from three-dimensional cubic lattice deposition simulations. (a) shows the dependence of ln(h) and ln(w) on ln(s). (b) shows the dependence of ln(N_s) on ln(s) where N_s is the number of clusters containing s sites.

completely inconsistent with an asymptotic value of 0.5 which might be expected for this model since the path from the top of a tree (or from the end of any of its branches) is a random walk to its base (Kondoh *et al* 1987, Takayasu and Nishikawa 1986). The scaling relationship $\tau = 2 - \nu_\parallel$ holds quite well for the 3D river network model. This is not surprising since the derivation of this relationship (Racz and Vicsek 1983) depends on the fact that the entire structure is compact but does not depend on the structure of its component trees.

There is already evidence indicating that the Eden models and ballistic deposition models belong to the same universality class (see Meakin *et al* 1986b, for example) in the sense that the exponents α, β and D are equal for these models. The results presented here lend further support to this idea. The two-dimensional simulations have been carried out on a large scale covering several orders of magnitude in length scales. In this case the evidence for universality is strong. The three-dimensional simulations have also been carried out on quite a large scale (particularly in the case of the lattice model). However, the range of length scales is substantially smaller and the uncertainties are correspondingly larger. In any event, the effective exponents obtained from these simulations satisfy quite well the scaling relationships $2\nu_\perp + \nu_\parallel = 1$ and $\tau = 1 - \nu_\parallel$.

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